A Study on Numerical Methods for Solving Differential Equations and Their Applications in Real World Modelling

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Abstract: This paper explores the application of numerical methods for solving differential equations, which are important tools for modelling multiple real world systems. In many cases, finding exact analytical solutions is not possible especially for complex or nonlinear problems. Therefore, numerical techniques are used to obtain accurate approximate solutions. Common methods such as Euler's Method, Improved Euler (heun's) Method, and Runge Kutta Method are addressed in this study. This paper focuses on the exponential and logistic growth models due to their importance in analyzing population behavior under different resource conditions.which are commonly used in population studies and in situations involving limited resources. By performing numerical imitation and comparing results this paper shows how these models can be effectively checked using numerical methods. Overall, the paper highlights the role of numerical techniques in connecting theoretical concepts with practical problem solving.

1. Introduction

Differential equations play a major role in mathematical modelling by describing how quantities change over time. They are often used in fields like physics, biology, economics and engineering to describe dynamic systems. But many real world differential equations are nonlinear or complex making analytical solutions difficult or impossible.

In such cases, numerical methods provide efficient approximate solutions through step by step computational techniques. Among the most commonly used methods are improved Euler's method, Euler's method, and Runge Kutta method, which provide a balance between clarity and high precision.

This paper implements these numerical methods to the Exponential and Logistic population growth models. The Exponential model shows perfect unrestricted growth, while the Logistic model includes carrying capacity for more realistic population performance. By analyzing the performance of different numerical methods this study shows the importance of numerical techniques in connecting mathematical theory and real world applications.

2. Methodology

The methodology used to investigate numerical methods for solving ordinary differential equations and their applications in population growth models is described in this part of the paper. The strategy follows a structured process that begins with model formulation and ends with the analysis of results.

2.1 Formulation of population growth models

The Exponential and Logistic population growth models are derived as first order ordinary differential equations representing population change over time. Ordinary idea are used to select initial conditions and model parameters.

2.2 Selection of numerical methods

Euler's Method, Improved Euler Method and the Fourth Order Runge Kutta Method are used to obtain numerical solutions. These methods are chosen due to their simplicity, accuracy, and common use in numerical analysis.

2.3 Numerical implementation

The numerical methods are applied to both models over a fixed time interval using an appropriate step size. Computations are carried out using MATLAB to obtain approximate solutions.

2.4 Comparison and analysis

The numerical solutions are presented graphically and compared to study accuracy and behavior of each method. Where analytical solutions are available, comparisons are made to evaluate approximation errors.

3. Numerical techniques for solving ordinary differential equation:

3.1. Euler's method One simple way to approximate the value of y_1 is to assume that the segment of the curve between P_0 and P_1 is a straight line with a slope equal to the gradient of the curve at P_0. Given a differential equation of the form $\frac{dy}{dx} = f(x, y)$

$$y_{r+1} = y_r + hf(x_r, y_r), \quad r = 0,1,2, \dots$$

3.2. Improved Euler Method: The Euler formula discussed earlier is based on the assumption that the slope of the line segment connecting two successive points, P_r and P_{r+1}, on the graph of $y(x)$ is equal to the value of $\frac{dy}{dx}$ at P_r. A more accurate approximation can be obtained by assuming that the slope between P_r and P_{r+1} is the average of the derivatives $\frac{dy}{dx}$ evaluated at both at points.

Based on this assumption, the Improved Euler (Heun's) method is given by

$$y_{r+1} = y_r + \frac{h}{2}\{f(x_r, y_r) + f(x_{r+1}, y_r + hf(x_r, y_r))\}$$

3.3. Runge-Kutta Method :

From Euler's formula, $y_{r+1} = y_r + hf(x_r, y_r)$

We denote $k_1 = hf(x_r, y_r)$

Therefore, proceeding with the approximation of $x_r + \frac{1}{2}h$

$$k_2 = hf(x_r + \frac{1}{2}h, y_r + \frac{1}{2}k_1),$$
$$k_3 = hf(x_r + \frac{1}{2}h, y_r + \frac{1}{2}k_2)$$
$$k_4 = hf(x_r + h, y_r + k_3),$$

$$y_{r+1} = y_r + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

4. Real world applications of logistic and exponential models:

4.1. The exponential growth model: This model was first proposed by Thomas R. Malthus in 1798 as a way to understand population dynamics. The central idea behind the model is that the rate at which a population grows is directly proportional to its current size. This means that as the population increases, the growth accelerates at a consistent rate leading to

exponential growth under ideal conditions. Malthus's model laid the groundwork for later studies on population growth and resource limitations.

Assumptions of the Model:

- .The population is sufficiently large, so it can be treated as a continuous quantity.
- .Population is homogenous that is, it is evenly spread over the living space.
- There are no limitations to growth resources such as food and space are unlimited. The population changes only due to births and deaths, with no immigration or emigration.

Formulation of the Model:

Let $x(t)$ denote the size of a population at time t where $x(0) = x_0$ is the initial population. Assume the population changes only due to births and deaths, with no immigration or emigration. Let $B(t)$ and $D(t)$ denote the total number of birth and deaths up time t, respectively .Then the per capita birth rate is denoted by b and the death rate by m.

$$b = \frac{1}{x(t)}\frac{dB}{dt}, \qquad\qquad m = \frac{1}{x(t)}\frac{dB}{dt}$$

and the per capita growth rate of the popuIation at any time t is given

$$\frac{1}{x(t)}\frac{d}{dt}(B - D) = b - m \qquad\qquad\qquad b - m =$$

$r \, (constant)$

$$\frac{dx(t)}{dt} = rx(t), \qquad\qquad where \; x(0) = x_0$$

(2)

Mathematical Calculations: Integrating this equation, we get (2)

$$\ln x(t) = rt + c , \qquad c = constant$$

To obtain c we use the initial condition $x(0) = x_0$ and get

$$\ln x_0 = c$$

(3)

Eqn.(3) in Eqn.(2) then reduce to $\; ln\frac{x(t)}{x_0} = rt = lne^{rt}$

Or, $$x(t) = x_0 e^{rt}$$

4.2. The logistic growth model: The logistic model explains how population growth is affected when resources are limited. In the beginning, the population increases rapidly, but over time, factors like limited food, space, and environmental resistance slow down the growth. Eventually, the population reaches a stable size where births and deaths balance out. This model was introduced by P.F. Verhulst in 1838 and provides a more realistic view of population dynamics compared to unlimited growth models.

Assumptions of the Model:

- When the population is small, it grows quickly.
- As the population gets larger, there is less space, food, and other resources.
- This slows down the growth and stops it when the population reaches a limit, called the carrying capacity (K).

Formulation of the Model: By considering r(x) as positive and applying $r(x) = r_1(1 - \frac{x}{K})$ in differential equation where constants $r_1 > 0 \ and \ k > 0$,we get Verhulst's famous logistic equation.

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{K}\right), \ x(0) = x_0 \tag{5}$$

Since $$r'(x) = -\frac{r_1}{K} < 0 \ for \ all \ x > 0$$

Mathematical Calculations:

Consider Equation (5)

$$\frac{dx}{dy} = r_1 x \left(1 - \frac{x}{K}\right), \qquad x(0) = x_0$$

This logistic equation can be solved using the method separation of variable.

$$\frac{Kdx}{x(K-x)} = r_1 dt \tag{6}$$

This can rewritten as: $$\left[\frac{1}{x} + \frac{1}{K-x}\right] dx = r_1 dt \tag{7}$$

If we assume that x<K, then Eqn. (7) on integration:

$$lnx - \ln(K - x) = r_1 t + c, \qquad c = constant \tag{8}$$

Using the initial condition $x(0) = x_0$, we obtain $c = ln\frac{x_0}{K-x_0}$ therefore,

$$ln\frac{x}{K-x} = lne^{r_1 t} + ln\frac{x_0}{K-x_0}$$

$$= ln\frac{x_0 e^{tr_1}}{K-x_0} \qquad (9)$$

$$\text{Or, } x[(K-x_0) + x_0 e^{tr_1}] = Kx_0 e^{tr_1}$$

$$x = \frac{K}{1+(\frac{K-x_0}{x_0})e^{-tr_1}} \qquad (10)$$

Therefore, $(t) = \frac{K}{1+c_1 e^{-rt}},$ $\qquad (c_1 = \frac{K-x_0}{x_0} \text{ is a constant })$

5. Graphical representation of exponential and logistic models using MATLAB.

5.1. Graphical Representation of Exponential Growth Model:

Case Study: Bacterial Growth in a Laboratory Culture

where Initial population : $x_0 = 150$, Growth rate : r =0.4 ,Time interval : t = 0 to t= 15

Using the exponential growth formula: $\qquad x(t) = x_0 e^{rt}$
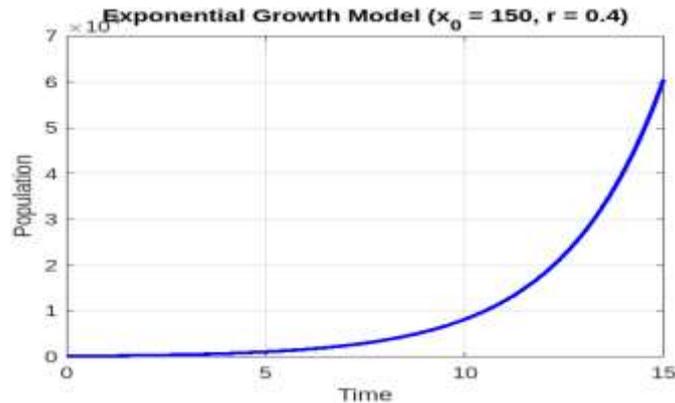
MATLAB code

```
>> % Exponential Growth Model

r = 0.4;              % Growth rate
x0 = 150;             % Initial population
t = 0:0.1:15;         % Time vector

x = x0 * exp(r * t);  % Exponential growth formula

plot(t, x, 'b', 'LineWidth', 2);
xlabel('Time');
ylabel('Population');
title('Exponential Growth Model (x_0 = 150, r = 0.4)');
grid on;
>> |
```

Output Graph:

Exponential Growth Model ($x_0 = 150$, $r = 0.4$)

Graph Explanation: The graph shows rapid continuous population growth from 150 characteristic of exponential growth with unlimited resources..

5.2. Graphical Representation of logistic Growth Model :

Case Study: Fish Population Growth in a Lake

Where Initial population: $x_0 = 200$, Growth rate: r= 0.5, Carrying capacity K=1000 ,Time interval t=0 to t=2

The logistic growth equation is: $x = \dfrac{K}{1+(\frac{K-x_0}{x_0})e^{-tr_1}}$
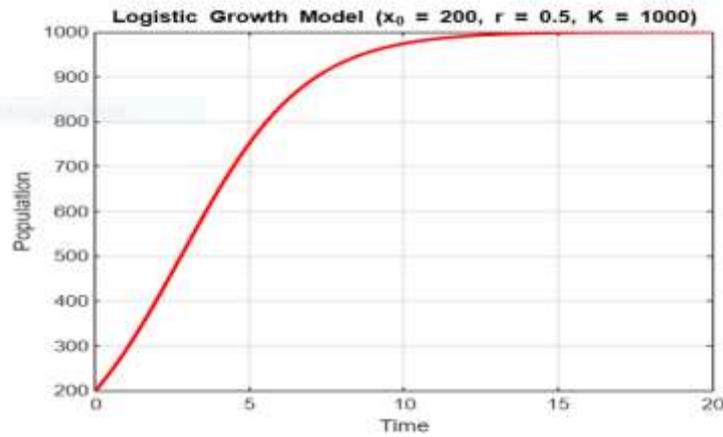
MATLAB Code:

```
>> % Logistic Growth Model

r = 0.5;              % Growth rate
x0 = 200;             % Initial population
K = 1000;             % Carrying capacity
t = 0:0.1:20;         % Time vector

x = K ./ (1 + ((K - x0)/x0) * exp(-r * t));   % Logistic formula

plot(t, x, 'r', 'LineWidth', 2);
xlabel('Time');
ylabel('Population');
title('Logistic Growth Model (x_0 = 200, r = 0.5, K = 1000)');
grid on;
```

Logistic Growth Model (x_0 = 200, r = 0.5, K = 1000)

Graph Explanation: The graph shows population rising rapidly at first then slowing and leveling off near the carrying capacity K=1000, reflecting resource limited growth.

5.3. Comparison of Exponential and Logistic Growth Models:

Parameters Used: $x_0 = 80$, r = 0.3, k = 800, t= 0 to t =25
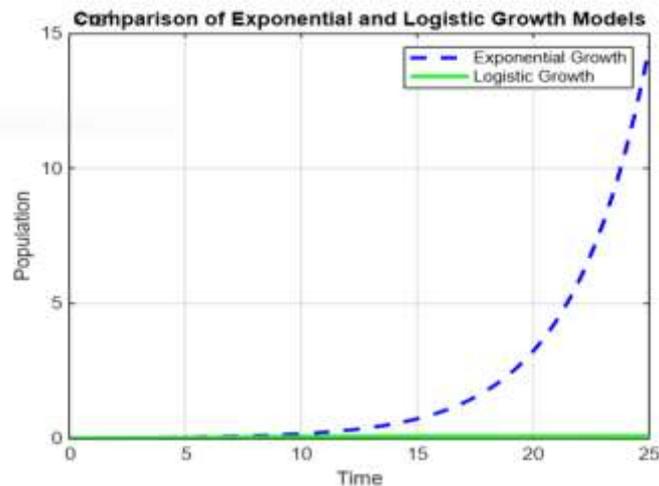
MATLAB Code:

```matlab
r = 0.3;
x0 = 80;
K = 800;
t = 0:0.1:25;

x_exp = x0 * exp(r * t);                    % Exponential growth
x_log = K ./ (1 + ((K - x0)/x0) * exp(-r * t));    % Logistic growth

plot(t, x_exp, 'b--', 'LineWidth', 2); hold on;
plot(t, x_log, 'g-', 'LineWidth', 2);
xlabel('Time');
ylabel('Population');
title('Comparison of Exponential and Logistic Growth Models');
legend('Exponential Growth', 'Logistic Growth');
grid on;
```

Output graph:

Graph Explanation: The graph compares exponential and logistic growth, both starting from a population of 80. The exponential curve rises sharply without limit, showing ideal conditions. The logistic curve at first shows rapid growth but progressively slows down and becomes stable near the carrying capacity K=800 showing real world limitations on available resources.

6. Conclusion

This paper point out the importance of differential equations in modelling dynamic systems in nature, technology, and society. Many real world equations are complex or nonlinear making exact solution difficult, so numerical methods like Euler's Method, Improved Euler's Method, and Runge Kutta Method are used for estimated solutions.

To show their purpose, exponential and logistic growth models were analyzed. The exponential model shows unlimited growth, while the logistic model accounts for carrying capacity making it more realistic. Applying these models in MATLAB provided clear insight exponential growth rises unrestricted, whereas logistic growth becomes stable due to resource limits.

Overall, this work shows that numerical methods properly bridge theory and real world application, providing accurate and computationally efficient solutions for complex differential equations.

7. References:

1. Arfken, G. Self-Adjoint Differential Equations. Mathematical methods for Physicists, 3rd ed.; Academic Press : Orlando, FL,USA, 1985

2. Atkinson K, Han W, Stewart D, (2009) Numerical Solution of Ordinary differential. New Jersey : John Wiley & sons, Hoboken

3. Byakatonda Denis, An overview of numerical and Analytical methods for solving Ordinary differential equations. University od kisubi

4. J. N. Sharma, 2004. Numerical methods for engineers and scientists

5. Nalin Dewan (2021). Population modelling: Exponential and logistic model. International research journal of engineering and technology.

6. Di Crescenzo A, Paraggio P. logistic growth described by Birth- death and diffusion processes. Mathematics. 2019

7. E Hairr, C. Lubich and G Wanner. Solving ordinary differential equation II. Stiff and differential algebraic problem, 2nd ed., Springer- Verlag, Berlin, 1996.

8. E. Isaacson and H. Keller. Analysis of numerical method, John Wiley, Newyork 1966.