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Improved estimators for mean of population using ancillary variable under linear systematic sampling

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Abstract: The information based on ancillary variable is essential for estimation of the mean of population in surveys related to the agriculture, medical and forest. Numerous ancillary variables information, including the correlation coefficient, coefficient of variation and kurtosis etc. might affect the estimators' performance. Many studies have been conducted using such information to estimate the mean of population under linear systematic sampling (LSS). There is a dearth of research on estimating the mean of population in LSS using the known coefficient of variation and intra-class correlation coefficient of the study variable. Thus, in the context of LSS, improved estimators for the mean of population have been suggested, utilizing the known coefficient of variation and intra-class correlation related to the study variable. These novel estimators' properties are examined, and the conditions under which they achieve better than their counterparts are determined. Furthermore, empirical investigation is done to evaluate the impact of the constructed estimators in comparison to the relevant estimators.

Keywords: Ancillary variable, Bias, MSE, Coefficient of Variation.

Mathematics Subject Classification 62D05 · 62G05 · 62H12

1. Introduction

Firstly, Madow and Madow introduced systematic sampling, has become a widely embraced method for surveying finite populations. Its application in different fields has been discussed by Finney (1948) and Sukhatme (1958). It is interesting because it can show concealed or obvious stratifications in a population, frequently with more accuracy than simple random sampling. This method also lowers survey expenses and streamlines deployment. Linear systematic sampling commences with the random selection of one unit, followed by the selection of consequent units according to a defined law.

In statistical research and sample surveys, the use of ancillary information, such as the known mean of population of an ancillary variable (X), is essential for improving the precision and effectiveness of estimators for population parameters. This ancillary information has been employed by Kushwaha et.al. (1989), Banarasi et al. (1993), Singh and Singh (1998) and Singh and Jatwa (2013) in the development of a number of estimators for mean of population in LSS.

By adding the study variable's known coefficient of variation, Searls (1964, 1967) developed improved estimators for the mean of population. Additionally, Khare and Kumar (2009, 11), Kumar and Kumar (2017) and Kumar and Zeeshan (2019) proposed many estimators for mean of population in the presence of non-response in simple random sampling (SRS) using the Searls estimator.

We have got limited research work from literature review for estimation of mean of population utilizing the study variable's known coefficient of variation and intra-class correlation under LSS. In light of this, we have suggested improved mean of population estimators in linear systematic sampling that use the study variable's known coefficient of variation and intra-class correlation coefficient. The suggested estimators' properties are examined. Further, comparative studies of the improved estimators have been made with the pertinent estimators. The empirical studies with two authentic data sets have been given in the support of the suggested estimators.

Remaining of the research paper begin with section 2, which describes the notations used in the research paper and earlier estimators, section 3 discuss about the suggested estimators and section 4 explain the mean square error (MSE) expression of the constructed and relevant estimators. In section 5, efficiency comparison of the constructed and relevant estimator is presented. Later in section 6, two data sets have been used to compare the suggested and relevant estimators. In result and conclusion, we have concluded from the empirical study that the constructed estimators do well than the relevant estimator. In the last, the derivations of the proposed estimators have been discussed.

2. **Notations and earlier estimators**

Let us suppose that Y is the study variable of interest and X is an ancillary variable in a population (finite) $U = (U_1, U_2, ..., U_N)$ that is defined in terms of units U_i (i=1…N). Let us envision that, $N = nk$, where N represents population size, n denotes the required sample size, and k corresponds to the sampling interval.

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Within the framework of LSS, let (y_{ij}, x_{ij}), i=1,2,…,k, j=1,2,…,n, represents the paired observations corresponding to the j^{th} unit in the i^{th} sample for the Y and X, respectively. Hence the LSS means for both the variables Y and X are defined as:

$$\bar{y}_{lss} = \frac{1}{n}\sum_{j=1}^{n} y_{ij} \text{ and } \bar{x}_{lss} = \frac{1}{n}\sum_{j=1}^{n} x_{ij} \;. \tag{2.1}$$

The respective variances of the corresponding LSS means $(\bar{y}_{lss}, \bar{x}_{lss})$ are acquired as:

$$V(\bar{y}_{lss}) = \bar{Y}^2\left[\left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_y\}\right]C_y^2$$

(2.2)

and

$$V(\bar{x}_{lss}) = \bar{X}^2\left[\left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_x\}\right]C_x^2 \;, \tag{2.3}$$

where $C_y = \frac{S_y}{\bar{Y}}, C_x = \frac{S_x}{\bar{X}}, S_y = \frac{1}{N-1}\sum_{j=1}^{N}(y_{ij} - \bar{Y}), S_x = \frac{1}{N-1}\sum_{j=1}^{N}(x_{ij} - \bar{X})$ and (ρ_y, ρ_x) are the respective correlation coefficient that represent the relation between paired units within the LSS for the Y and X.

For estimation of mean of population in LSS, the ratio (T_1) and product (T_2) estimators using the known mean of population of X, have been discussed by Swain (1964) and Shukla (1971) as:

$$T_1 = \bar{y}_{lss}\frac{\bar{X}}{\bar{x}_{lss}} \text{ and } T_2 = \bar{y}_{lss}\frac{\bar{x}_{lss}}{\bar{X}} \;. \tag{2.4}$$

Further, Singh and Singh (1998) have proposed generalised estimator for mean of population which is given below.

$$T_3 = \bar{y}_{lss}\left(\frac{\bar{x}_{lss}}{\bar{X}}\right)^{\beta} , \tag{2.5}$$

where β is a constant.

3. Suggested Estimators

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Searls (1964) first introduced the estimator for mean of population utilizing the known coefficient of variation of Y in simple random sampling without replacement, which is deliberated below as:

$$\bar{y}_s = \lambda_o \bar{y}_{srs}, \tag{3.1}$$

where $\bar{y}_{srs} = \frac{1}{n}\sum_{l=1}^{n} y_l$ and λ_o is constant.

The MSE of the estimator $\bar{y}_s$ is acquired as:

$$MSE(\bar{y}_s) = \bar{Y}^2\left[(\lambda_o - 1)^2 + \lambda_o^2 \gamma_o C_y^2\right], \tag{3.2}$$

where $\gamma_o = \left(\frac{1}{n} - \frac{1}{N}\right)$.

The optimum value of λ_o is determined by minimizing estimator's MSE, as shown below:

$$\lambda_{o(opt)} = \left(1 + \gamma_o C_y^2\right)^{-1}. \tag{3.3}$$

Using the optimum value of λ_o from (3.3), we acquire the minimum MSE of estimator $\bar{y}_s$ in simple random sampling as:

$$MSE(\bar{y}_s)_{\min} = (1 - 2D_o)\gamma_o S_y^2,$$

where $D_o = \gamma_o C_y^2$.

By utilizing the known coefficient of variation and intra-class correlation coefficient of study variable, the estimator for mean of population under LSS is defined as:

$$\bar{y}^* = \lambda_1 \bar{y}_{lss}, \tag{3.4}$$

where λ_1 is constant.

The MSE of the estimator $\bar{y}^*$ is acquired as:

$$MSE(\bar{y}^*) = \bar{Y}^2\left[(\lambda_1 - 1)^2 + \lambda_1^2 \gamma_1 C_y^2\right], \tag{3.5}$$

where $\gamma_1 = \left(\frac{N-1}{Nn}\right)\left\{1 + (n-1)\rho_y\right\}$.

Differentiating equation (3.5) with respect to λ_1 and putting it equal to 0, we have got the value (optimum) of λ_1 as:

$$\lambda_{1(opt)} = (1 + D)^{-1}, \tag{3.6}$$

where $D = \left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_y\}C_y^2$.

By putting the value of λ_1 into equation (3.5), we acquire the minimum MSE of $\bar{y}^*$ as follows:

$$MSE(\bar{y}^*)_{\min} = (1-2D)\gamma_1 S_y^2. \tag{3.7}$$

By incorporating $\bar{y}^*$ and the known population mean $\bar{X}$ of the ancillary variable *x*, we have developed ratio, product, and generalized ratio estimators for the mean of population in LSS. A detailed discussion of these estimators is provided below.

$$T_4 = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}_{lss}}\right), T_5 = \bar{y}^*\left(\frac{\bar{x}_{lss}}{\bar{X}}\right) \quad \text{and} \quad T_6 = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}_{lss}}\right)^{\alpha}, \tag{3.8}$$

where α is constant.

For $\alpha = 0$, $\alpha = 1$ and $\alpha = -1$, T_6 reduce to $\bar{y}^*$, T_4 and T_5 respectively.

The known coefficient of variation and intra class correlation coefficient of study variable that are utilized in the constructed estimator, can be found from previous data, as recommended by Reddy (1978) and Tripathi et al. (1983).

Conversely, these values can also be utilized from sample values which have no influence on the MSE of the constructed estimators up to the term of order $1/n$, as recommended by Srivastava and Jhajj (1983).

4. Mean square error (MSE) of the constructed estimators

The MSE of the proposed estimators T_4, T_5 and T_6 are derived up to the term of order $1/n$ as:

$$MSE(T_4) = \bar{Y}^2\left[(1-2D)\gamma_1 C_y^2 + (1-4D)\gamma_2 C_x^2 - 2(1-3D)\gamma_3 C_{yx}\right], \tag{4.1}$$

$$MSE(T_5) = \bar{Y}^2\left[(1-2D)\gamma_1 C_y^2 + (1-2D)\gamma_2 C_x^2 + 2(1-3D)\gamma_3 C_{yx}\right] \tag{4.2}$$

and

$$MSE(T_6) = \bar{Y}^2\left[(1-2D)\gamma_1 C_y^2 + \alpha(\alpha-(3\alpha+1)D)\gamma_2 C_x^2 - 2\alpha(1-3D)\gamma_3 C_{yx}\right], \tag{4.3}$$

where $C_{yx} = \frac{S_{yx}}{\bar{Y}\bar{X}}$, $\gamma_1 = \left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_y\}$, $\gamma_2 = \left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_x\}$,

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$$\gamma_3 = \left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_y\}^{\frac{1}{2}}\{1+(n-1)\rho_x\}^{\frac{1}{2}}.$$

By differentiating equation (4.3) with respect to α and putting it equal to 0, we derive the value (optimum) of α as:

$$\alpha_{(opt)} = \frac{\gamma_3 C_{yx}}{\gamma_2 C_x^2} + \frac{D}{2(1-3D)}. \quad (4.4)$$

Hence, the minimum MSE of estimator T_6 is gotten as:

$$MSE(T_6)_{\min} = \bar{Y}^2\left[(1-2D)\gamma_1 C_y^2 - \left(\frac{(1-3D)(\gamma_3 C_{yx})^2}{\gamma_2 C_x^2} + \frac{D^2\gamma_2 C_x^2}{4(1-3D)} + D\gamma_3 C_{yx}\right)\right].$$

(4.5)

The MSE's expression for the relevant estimators (T_1, T_2, T_3) are acquired as follows:

$$MSE(T_1) = \bar{Y}^2\left[\gamma_1 C_y^2 + \gamma_2 C_x^2 - 2\gamma_3 C_{yx}\right], \quad (4.6)$$

$$MSE(T_2) = \bar{Y}^2\left[\gamma_1 C_y^2 + \gamma_2 C_x^2 + 2\gamma_3 C_{yx}\right], \quad (4.7)$$

$$MSE(T_3) = \bar{Y}^2\left[\gamma_1 C_y^2 - \frac{\gamma_3 C_{yx}}{\gamma_2 C_x^2}\right]. \quad (4.8)$$

5. Efficiency Comparison

Under the definite conditions, the proposed estimators (T_4, T_5, T_6) exhibit lower MSE compared to ($\bar{y}_{lss}, \bar{y}^*$) and other relevant estimators (T_1, T_2, T_3) as outlined below.

$$MSE(T_4) < Var(\bar{y}_{lss}) \text{ If } \rho_{yx} > \frac{2D\gamma_1 C_y^2 - (1-4D)\gamma_2 C_x^2}{2(1-3D)C_y C_x} \quad (5.1)$$

$$MSE(T_4) < Var(\bar{y}^*) \text{ If } \rho_{yx} > \frac{(1-4D)}{2(1-3D)\gamma_1 C_y^2} \quad (5.2)$$

$$MSE(T_4) < MSE(T_1) \text{ If } \rho_{yx} < \frac{(\gamma_1 C_y^2 + 2\gamma_2 C_x^2)}{3C_y C_x} \quad (5.3)$$

$$MSE(T_5) < Var(\bar{y}_{lss}) \text{ If } \rho_{yx} > \frac{(1-2D)\gamma_2 C_x^2 - 2D\gamma_1 C_y^2}{2(1-3D)C_y C_x} \quad (5.4)$$

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$$MSE(T_5) < Var(\bar{y}^*) \text{ If } \rho_{yx} < -\frac{(1-2D)}{2(1-3D)\gamma_1 C_y^2} \tag{5.5}$$

$$MSE(T_5) < MSE(T_2) \text{ If } \rho_{yx} > -\frac{(\gamma_1 C_y^2 + \gamma_2 C_x^2)}{3C_y C_x} \tag{5.6}$$

$$MSE(T_6) < Var(\bar{y}_{lss}) \text{ If } \rho_{yx} > \frac{\alpha(\alpha-(3\alpha+1)D)\gamma_2 C_x^2 - 2D\gamma_1 C_y^2}{2\alpha(1-3D)C_y C_x} \tag{5.7}$$

$$MSE(T_6) < Var(\bar{y}^*) \text{ If } \rho_{yx} > \frac{(\alpha-(3\alpha+1)D)}{2(1-3D)\gamma_1 C_y^2} \tag{5.8}$$

$$MSE(T_6) < MSE(T_3) \text{ If } \rho_{yx} < \frac{D(2\gamma_1 C_y^2 + \alpha(3\alpha+1)\gamma_2 C_x^2)}{2\alpha(3D-2)C_y C_x} \tag{5.9}$$

6. Empirical study

6.1 *Data Set 1*

This dataset sourced from Sarndal et al. (1992) and referenced in Appendix B. In this dataset, Real estate value according to 1984 and Revenues from the 1985 municipal council are considered as Y and X respectively. The corresponding parameter values are provided below.

$N = 252, n = 21, \bar{Y} = 3151.02, \bar{X} = 253.8452, S_y = 4982.834, S_x = 629.5168, \rho_y = -0.0227,$

$\rho_x = -0.021, \rho_{yx} = 0.936$.

6.2 *Data Set 2*

This dataset, sourced from Murthy (1967). In this dataset, the length of the timber serves as Y and the volume of the timber serves as X .The parameter values are provided below.

$N = 176, n = 16, \bar{Y} = 282.61, \bar{X} = 6.99, S_y = 155.73, S_x = 2.95, \rho_y = -0.0019, \rho_x = -0.0014, \rho_{yx} = 0.87$

The proposed estimators' performance is checked by calculating percentage relative efficiency (PRE). It is frequently utilized to show how more or less efficient the estimator is in comparison to the other estimators. The PRE of the estimators can be computed by applying the subsequent formula as:

$$PRE(T) = \frac{MSE(\bar{y}_{lss})}{MSE(T)} \times 100, \tag{6.1}$$

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where $T = T_1, T_2, T_3, T_4, T_5, T_6$ and $\bar{y}_{lss}$.

Table 1: MSE and PRE of the estimator with respect to $\bar{y}_{lss}$

Estimator	Data Set 1		Data Set 2	
	MSE	PRE	MSE	PRE
$\bar{y}_{lss}$	642982.80	100.00	1464.17	100.00
$\bar{y}^*$	559705.80	114.87	1410.49	103.81
T_1	376235.20	170.90	370.93	394.73
T_2	4269356.00	015.06	4288.36	034.14
T_3	79066.15	813.22	355.94	411.35
T_4	235998.30	272.45	361.50	405.02
T_5	3590345.00	017.91	4095.23	035.75
T_6	40130.15	1602.24	345.18	424.18

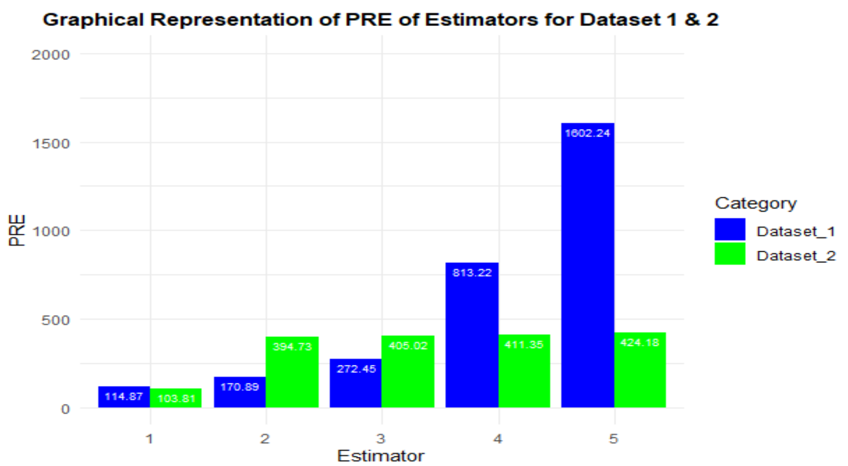

Figure 1: The PRE of the estimators 1 ($\bar{y}$), 2 (T_1), 3 (T_4), 4 (T_3), 5 (T_6)

A bar graph of PRE is presented in figure 1, In this bar graph the comparison of PRE of Data set 1 and 2 has been shown.

7. Results and conclusion

Table 1 presents the MSE and PRE values of both the suggested and relevant estimators for two data sets. Analysis of data sets 1 and 2 reveals that, for each distinct population, the estimators (T_4, T_6) exhibit lower MSE compared to the estimators (T_1, T_3) and other pertinent estimators ($\bar{y}_{lss}, \bar{y}^*$). Additionally, estimator T_6 has been found to produce a lower MSE in contrast to estimators (T_4, T_5). However, some estimators (T_2, T_5) show higher MSE than ($\bar{y}_{lss}, \bar{y}^*$) due to their failure to satisfy the essential conditions for improved performance. Hence, we can draw conclusion that the built estimators are more effective in contrast to the other present estimators, as they incorporate ancillary information on coefficient of variation and intra-class correlation coefficient of study variable. Consequently, these estimators are recommended for estimation of the mean of population under LSS in surveys associated to agriculture, medical and forest.

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Declarations

Conflict of Interest The researcher declare that no conflict of interest exist

Competing Financial Interest The authors affirm that they have no financial stakes or personal affiliations that might have influenced the work presented in this paper.

Ethical Statement Since the study is based on already published data there are no human/animal subject in this work so an ethics statement is not valid.

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Appendix

To attain the MSE's expressions of the estimators under large sample approximation, let

$\bar{y} = \bar{Y}(1+\varepsilon_o)$, $\bar{x} = \bar{X}(1+\varepsilon_1)$ such that $E(\varepsilon_o) = E(\varepsilon_1) = 0$

$$E(\varepsilon_o^2) = \frac{1}{\bar{Y}^2}\left[\left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_y\}S_y^2\right], E(\varepsilon_1^2) = \frac{1}{\bar{X}^2}\left[\left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_x\}S_x^2\right]$$

and

$$E(\varepsilon_o\varepsilon_1) = \frac{1}{\bar{Y}\bar{X}}\left[\left(\frac{N-1}{Nn}\right)\{1+(n-1)\rho_y\}^{\frac{1}{2}}\{1+(n-1)\rho_x\}^{\frac{1}{2}}S_{yx}\right].$$

We have

$$\bar{y}^* = \lambda_1\bar{y}$$

$$= \lambda_1\bar{Y}(1+\varepsilon_0)$$

$$MSE(\bar{y}^*) = E(\bar{y}^* - \bar{Y})$$

$$= E(\lambda_1\bar{Y}(1+\varepsilon_0) - \bar{Y})^2$$

$$= \bar{Y}^2\left[(\lambda_1 - 1)^2 + \lambda_1^2 E(\varepsilon_o^2) + 2\lambda_1(\lambda_1 - 1)E(\varepsilon_o)\right]$$

$$= \bar{Y}^2\left[(\lambda_1 - 1)^2 + \lambda_1^2 E(\varepsilon_o^2)\right]$$

$$MSE\left(\bar{y}^*\right)=\bar{Y}^2\left[\left(\lambda_1-1\right)^2+\lambda_1^2\gamma_1C_y^2\right], \tag{A.1}$$

where $\gamma_1=\left(\frac{N\text{ -1}}{Nn}\right)\left\{1+\left(n-1\right)\rho_y\right\}$.

By differentiating equation (A.1) with respect to λ_1 and equating it to zero, we derive the optimum value of λ_1 as:

$$\lambda_{1(opt)}=\left(1+D\right)^{-1}, \tag{A.2}$$

where $D=\left(\frac{N-1}{Nn}\right)\left\{1+\left(n-1\right)\rho_y\right\}C_y^2$.

Hence, the minimum MSE of estimator $\bar{y}^*$ is gotten as:

$$MSE\left(\bar{y}^*\right)_{\min}=\left(1-2D\right)\gamma_1S_y^2. \tag{A.3}$$

We have

$$T_6=\lambda_1\bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right)^\alpha$$

$$T_6=\lambda_1\bar{Y}\left(1+\varepsilon_o\right)\left(\frac{\bar{X}}{\bar{X}\left(1+\varepsilon_1\right)}\right)^\alpha$$

$$T_6=\lambda_1\bar{Y}\left(1+\varepsilon_o\right)\left(1+\varepsilon_1\right)^{-\alpha}$$

$$T_6=\lambda_1\bar{Y}\left(1+\varepsilon_o\right)\left(1-\alpha\varepsilon_1+\frac{\alpha\left(\alpha+1\right)\varepsilon_1^2}{2}...\right)$$

Afterwards avoiding the terms of $\varepsilon's$ up to the 2nd degree, we get

$$T_6=\lambda_1\bar{Y}\left(1+\varepsilon_o-\alpha\varepsilon_1-\alpha\varepsilon_o\varepsilon_1+\frac{\alpha\left(\alpha+1\right)\varepsilon_1^2}{2}\right).$$

The estimator's MSE is derived as follows:

$$MSE\left(T_6\right)=E\left[T_6-\bar{Y}\right]^2$$

$$MSE\left(T_6\right)=\mathrm{E}\left[\lambda_1\bar{Y}\left(1+\varepsilon_o-\alpha\varepsilon_1-\alpha\varepsilon_o\varepsilon_1+\frac{\alpha\left(\alpha+1\right)\varepsilon_1^2}{2}\right)-\bar{Y}\right]^2$$

$$MSE(T_6) = \bar{Y}^2 \left[\lambda_1^2 E\left(1 + \varepsilon_o^2 + \alpha^2 \varepsilon_1^2 + 2\varepsilon_o - 2\alpha\varepsilon_1 - 2\alpha\varepsilon_o\varepsilon_1 + \frac{2\alpha(\alpha+1)\varepsilon_1^2}{2} - 2\alpha\varepsilon_o\varepsilon_1\right)\right.$$

$$\left. + 1 - 2\lambda_1 E\left(1 + \varepsilon_o - \alpha\varepsilon_1 - \alpha\varepsilon_o\varepsilon_1 + \frac{\alpha(\alpha+1)\varepsilon_1^2}{2}\right)\right]$$

$$MSE(T_6) = \bar{Y}^2 \left[\lambda_1^2 E\left(1 + \varepsilon_o^2 + \alpha^2 \varepsilon_1^2 - 4\alpha\varepsilon_o\varepsilon_1 + \frac{2\alpha(\alpha+1)\varepsilon_1^2}{2}\right) + 1 - 2\lambda_1 E\left(1 - \alpha\varepsilon_o\varepsilon_1 + \frac{\alpha(\alpha+1)\varepsilon_1^2}{2}\right)\right]$$

By simplifying the above expression, we get

$$MSE(T_6) = \bar{Y}^2 \left[\lambda_1^2 + 1 - 2\lambda_1 + \lambda_1^2 E\left(\varepsilon_o^2\right) + \alpha E\left(\varepsilon_1^2\right)\left(\lambda_1^2\alpha + \lambda_1^2(\alpha+1) - \lambda_1(\alpha+1)\right) + 2\alpha E(\varepsilon_o\varepsilon_1)\left(\lambda_1 - 2\lambda_1^2\right)\right]$$

By putting the optimum value of λ_1 from equation (A.2), we get

$$MSE(T_6) = \bar{Y}^2 \left[(1-2D)E\left(\varepsilon_o^2\right) + \alpha(\alpha - (3\alpha+1)D)E\left(\varepsilon_1^2\right) - 2\alpha(1-3D)E(\varepsilon_o\varepsilon_1)\right]$$

$$MSE(T_6) = \bar{Y}^2 \left[(1-2D)\gamma_1 C_y^2 + \alpha(\alpha - (3\alpha+1)D)\gamma_2 C_x^2 - 2\alpha(1-3D)\gamma_3 C_{yx}\right]. \quad (A.4)$$

The MSE expressions of T_4 and T_5 can be derived by putting α = 1 and -1 in equation (A.4).